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EQUIVALENT INCLUSION METHOD FOR THE WORK-HARDENING BEHAVIOR OF PIEZOELECTRIC COMPOSITES

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Abstract—The work-hardening behavior of piezoelectric composites is investigated by utilizing the variational principle and the equivalent inclusion method. When the matrix of a piezoelectric composite subjected to a uniaxial tension is perfectly plastic and the piezoelectric inhomogeneities are elastic, the linear work-hardening rate is predicted. In particular, when both the matrix and the inhomogeneities of the piezoelectric material are transversely isotropic with different electroelastic moduli, and shapes of the inhomogeneities are elliptical, ribbon-like, rod-shaped and penny-shaped, the yielding stress and the work-hardening rate are obtained in closed forms. The results show that the work-hardening rate is proportional to the volume fraction of the inhomogeneities. Moreover, it is found that the work-hardening rate and the yielding stress are dependent on the shape and the volume fraction of the inhomogeneities but are independent on the size of the inhomogeneities.

1. INTRODUCTION

For years piezoelectric materials have served as the heart of many transducers and sensors with the responsibility of electromechanical energy conversion. They are typically utilized for generating an acoustic pulse from an input external loading (or electric signal) and detecting an acoustic pulse and then converting it into an electric signal (Dunn and Taya, 1993). These kinds of materials are referred to as *smart* or *active materials* which can feedback the internal states of a material or structure. This may explain why piezoelectric materials constitute an important branch of the recently emerging technologies of modern engineering materials.

Historically, improvement in piezoelectric materials and other applications centered on the development of improved single phase materials. Recently considerable research has been directed toward the development of piezoelectric composite materials which use the advantageous properties of two or more materials in combination. Furthermore, it is reported experimentally that piezoelectric composite materials work-harden much faster than do those consisting of a single phase. This is because piezoelectric composite materials do not deform equally. If piezoelectric inclusions (such as fiber, precipitate, dispersed particle, etc.) have larger elastic limits than that of the matrix, plastic deformations are restricted in the matrix, and the plastic flow in the matrix is constrained by the inclusions. Therefore, the deformation gradient is increased. The plastic deformation of piezoelectric composites with high strength inclusions becomes a very important problem in materials engineering and stimulates researchers to establish a theoretical framework for such a problem.

To the author's knowledge, no such framework exists at this time. The purpose of this work is to develop such a theory to explain the work-hardening behavior of piezoelectric composites subjected to a uniaxial tension. The approach of Mura (1987) for isotropic materials is generalized to the inherently anisotropic coupled behavior of piezoelectric heterogeneous media. By means of the variational principle (Huang and Mura, 1994) and the equivalent inclusion method (Eshelby, 1957), the coupling work-hardening behavior of piezoelectric composite materials between the mechanical and electric field is investigated. Special attention is focused on when both of the piezoelectric materials of matrix and inhomogeneities are transversely isotropic with different electroelastic constants, and shapes of the inhomogeneities are elliptical, ribbon-like, rod-shaped, and penny-shaped.

2. VARIATIONAL METHOD

Before proceeding, a few notations used in the subsequent sections are introduced. The usual summation convention applies to repeated Latin subscripts throughout this paper. Lowercase Latin subscripts range from 1 to 3, while uppercase subscripts range from 1 to 4. For example, $T_J U_J = T_j U_j + T_4 U_4$, where $j = 1$–3. Denote the elastic, piezoelectric and dielectric moduli of a piezoelectric material as (Barnett and Lothe, 1975)

$$\hat{C}_{iJKl} = \begin{cases} C_{ijkl} & J, K = 1,2,3 \\ e_{lij} & J = 1,2,3; \quad K = 4 \\ e_{ikl} & J = 4; \quad K = 1,2,3 \\ -\kappa_{il} & J, K = 4 \end{cases}, \tag{1}$$

where C_{ijkl} are elastic moduli measured at a constant electric field, e_{ikl} is the piezoelectric coefficient measured at a constant strain or electric field, and κ_{il} is the dielectric constant measured at a constant strain. In such shorthand notation, the elastic displacement–electric potential, U_J, the elastic strain–electric potential gradient, Z_{Ji}, and the stress–electric displacement, Σ_{iJ}, can be combined to get the following forms:

$$U_J = \begin{cases} u_j & J = 1,2,3 \\ \phi & J = 4 \end{cases}, \tag{2a}$$

$$Z_{Ji} = \begin{cases} \varepsilon_{ji} & J = 1,2,3 \\ -E_i = \phi_{,i} & J = 4 \end{cases}, \tag{2b}$$

$$\Sigma_{iJ} = \begin{cases} \sigma_{ij} & J = 1,2,3 \\ D_i & J = 4 \end{cases}. \tag{2c}$$

Here u_j, ε_{ji}, σ_{ij}, E_i, ϕ, and D_i represent the elastic displacement, strain, stress, electric field, electric potential, and electric displacement, respectively.

Next, consider a piezoelectric medium D which consists of matrix D_0 with the elastic, piezoelectric and dielectric moduli $\hat{C}_{iJMn}$ and inhomogeneities D_1 with the elastic, piezoelectric and dielectric moduli $\hat{C}^*_{iJMn}$. The domain D_1 denotes the total volume occupied by many ellipsoidal inhomogeneities. The possible mechanical state is determined by the Hamilton's principle which holds for irreversible processes under some assumptions necessary for an individual problem. The principle states that a variation of the Lagrangian δL is equal to the dissipation δQ, i.e.

$$\delta L = \delta Q. \tag{3}$$

Suppose a surface traction and a normal electric displacement, t_J, are prescribed on the boundary S of D. In the absence of body forces and free electric charges in a piezoelectric solid, the variation of the Lagrangian δL for a static case is expressed as

$$\delta L = \int_S t_J(\delta U_J^0 + \delta U_J)\,\mathrm{d}S - \int_D (\Sigma_{iJ}^0 + \Sigma_{iJ})(\delta U_{J,i}^0 + \delta U_{J,i} - \delta Z_{Ji}^{\mathrm{p}})\,\mathrm{d}V, \tag{4}$$

where U_J^0 is caused by the uniform applied load σ_{ij}^0 and the prescribed electric displacement D_i^0 when the material has no inhomogeneity and no plastic strain ($Z_{Ji}^{\mathrm{p}} = 0$), while U_J and

Σ_{iJ} refer to the quantities induced by the presence of the inhomogeneity and plastic strain Z^{p}_{Ji}. When the equations of equilibrium and boundary conditions

$$\Sigma^{0}_{iJ,i} = 0, \quad \Sigma_{iJ,i} = 0 \quad \text{in } D, \tag{5a}$$

$$\Sigma^{0}_{iJ}n_i = t_J, \quad \Sigma_{iJ}n_i = 0 \quad \text{on } S, \tag{5b}$$

are satisfied, eqn (4), after integration by parts, becomes

$$\delta L = \int_D (\Sigma^{0}_{iJ} + \Sigma_{iJ})\delta Z^{p}_{Ji}\,\mathrm{d}V. \tag{6}$$

In the present study, the matrix of the piezoelectric composite material is assumed to be perfectly plastic with yielding stress σ_y and the piezoelectric inhomogeneities are plastically nondeforming. The piezoelectric composite is subjected to a uniaxial tension $\sigma^{0}_{33} = \sigma^{0}$ in the x_3 direction (other components $\sigma^{0}_{ij} = 0$ and the electric displacement $D^{0}_{i} = 0$), and the piezoelectric composite material behaves like a work-hardening material. Since the matrix is a perfectly plastic material, the plastic strain ε in the tensile direction and $-\varepsilon/2$ in the perpendicular direction are occurring in tensile test. The yielding stress is σ_y and the dissipation δQ can be written as

$$\delta Q = \int_{D_0} \sigma_y \delta\varepsilon\,\mathrm{d}V. \tag{7}$$

Substituting eqns (6) and (7) into eqn (3) leads to

$$D_0\sigma^{0}\delta\varepsilon + \int_{D_0} \sigma_{ij}\delta\varepsilon^{p}_{ji}\,\mathrm{d}V = D_0\sigma_y\delta\varepsilon, \tag{8}$$

since only plastic strain ε^{p}_{ji} exists. Rewrite the integral term in the last equation as

$$\int_{D_0} \sigma_{ij}\delta\varepsilon^{p}_{ji}\,\mathrm{d}V = \int_{D-D_1} \sigma_{ij}\delta\varepsilon^{p}_{ji}\,\mathrm{d}V = -\int_{D_1} \sigma_{ij}\delta\varepsilon^{p}_{ji}\,\mathrm{d}V. \tag{9}$$

In derivation of the equation above, the following identity has been employed:

$$\int_D \sigma_{ij}\delta\varepsilon^{p}_{ji}\,\mathrm{d}V = \int_D \sigma_{ij}\delta u^{p}_{j,i}\,\mathrm{d}V = \int_S \sigma_{ij}\delta u^{p}_{j}n_i\,\mathrm{d}V - \int_D \sigma_{ij,i}\delta u^{p}_{j}\,\mathrm{d}V = 0 \tag{10}$$

where the last two integrals in the preceding equation vanish due to $\sigma_{ij}n_j = 0$ on S and $\sigma_{ij,j} = 0$ in D. Equation (9) reveals that a uniform plastic strain ε^{p}_{ji} defined in the matrix $D - D_1$ is equivalent to a uniform plastic strain $-\varepsilon^{p}_{ji}$ defined in the domain D_1.

If the distance between inhomogeneities is far enough to neglect the interaction, the elastic stress and the electric displacement, Σ_{iJ}, in D_1 becomes constant when the inhomogeneities are ellipsoidal (Huang and Yu, 1994). Therefore, eqn (9) can be evaluated as

$$-\int_{D_1} \sigma_{ij}\delta\varepsilon^{p}_{ji}\,\mathrm{d}V = -D_1(\sigma_{33}\delta\varepsilon - \sigma_{11}\delta\varepsilon/2 - \sigma_{22}\delta\varepsilon/2). \tag{11}$$

Finally, eqn (8) becomes

$$\sigma^0 - f(\sigma_{33} - \sigma_{11}/2 - \sigma_{22}/2) = \sigma_y, \tag{12}$$

where $f = D_1/D_0 \approx D_1/D$ is the volume fraction of inhomogeneities. The uniform internal stress σ_{ij} can be determined by applying the equivalent inclusion method (Eshelby, 1957) as shown in the following section.

3. EQUIVALENT INCLUSION METHOD

Let the inhomogeneity be an ellipsoid defined by

$$\Omega : \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \leqslant 1, \tag{13}$$

where a_1, a_2 and a_3 are the lengths of the semiaxis of the ellipsoid. The plastic strain $-Z^{\text{p}}_{Kl}$ inside the domain Ω will induce elastic displacement and electric potential, U_K, elastic strain and electric field, Z_{Kl}, and elastic stress and electric displacement, Σ_{iJ}. The induced stress and electric displacement can be calculated by applying the equivalent inclusion method which states that the presence of an inhomogeneity can be simulated by an inclusion in the homogeneous material with plastic strain $-Z^{\text{p}}_{Kl}$ plus the equivalent eigenstrain Z^*_{Kl}, i.e.

$$\begin{aligned}\Sigma^0_{iJ} + \Sigma_{iJ} &= \hat{C}^*_{iJKl}(Z^0_{Kl} + Z_{Kl} + Z^{\text{p}}_{Kl}) \\ &= \hat{C}_{iJKl}(Z^0_{Kl} + Z_{Kl} + Z^{\text{p}}_{Kl} - Z^*_{Kl}) \quad \text{in } D_1.\end{aligned} \tag{14}$$

It is seen that Σ_{iJ} is equivalent to the elastic stress and electric displacement caused by $Z^*_{Kl} - Z^{\text{p}}_{Kl}$ uniformly distributed inside the domain D_1 assuming that the whole domain D is homogeneous with the same electroelastic constants $\hat{C}_{iJKl}$ as those of the matrix. The induced strain and electric field, Z_{Kl}, can simply be obtained for an ellipsoidal inclusion as

$$Z_{Kl} = \mathbf{S}_{KlAb}(Z^*_{Ab} - Z^{\text{p}}_{Ab}), \tag{15}$$

where $\mathbf{S}_{KlAb}$ is referred to as Eshelby's tensor for anisotropic material of piezoelectricity and has the properties below

$$\mathbf{S}_{KlAb} = \begin{cases} S_{klab} = S_{lkab} = S_{lkba} = S_{klba} & K, A = 1,2,3 \\ S_{kl4b} = S_{lk4b} & K = 1,2,3\,;\, A = 4 \\ S_{4lab} = S_{4lba} & K = 4\,;\, A = 1,2,3 \\ S_{4l4b} & K, A = 4 \end{cases}. \tag{16}$$

On substituting eqn (15) into eqn (14), it follows that

$$h_{iJMn} Z^*_{Mn} = H_{iJ}, \tag{17}$$

where

$$h_{iJMn} = [(\hat{C}^*_{iJAb} - \hat{C}_{iJAb}) S_{AbMn} + \hat{C}_{iJMn}], \tag{18a}$$

$$H_{iJ} = L_{iJnM}\Sigma^0_{nM} - T_{iJMn} Z^{\text{p}}_{Mn}, \tag{18b}$$

and

$$I_{iJnM} = I_{iJnM} - \hat{C}^*_{iJAb}\hat{C}^{-1}_{AbnM}, \tag{19a}$$

$$T_{iJMn} = (\hat{C}^*_{iJAb} - \hat{C}_{iJAb})(S_{AbMn} - I_{AbMn}), \tag{19b}$$

$$I_{AbMn} = \begin{cases} (\delta_{am}\delta_{bn}+\delta_{bm}\delta_{an})/2 & A, M \leqslant 3 \\ \delta_{bn} & A = M = 4 \\ 0 & \text{otherwise.} \end{cases} \tag{19c}$$

Equations (18) and (19) are determined when the electroelastic constants and the shape of the inclusion are specified.

Since the piezoelectric composite is subject to a uniaxial tension in the x_3 direction, $\Sigma^0_{iJ} = 0$ except $\Sigma^0_{33} = \sigma^0$, and $\hat{C}^{-1}_{AbnM}\Sigma^0_{nM} = \hat{C}^{-1}_{Ab33}\sigma^0$. In addition, to represent h_{iJMn} and H_{iJ} in a 9×9 matrix and 9×1 column vectors, respectively, the following mapping of adjacent indices is utilized (Dunn and Taya, 1993):

$$\begin{aligned} &11 \to 1, \quad 22 \to 2, \quad 33 \to 3, \quad 23 \to 4, \quad 13 \to 5, \\ &12 \to 6, \quad 14 \to 7, \quad 24 \to 8, \quad 34 \to 9. \end{aligned} \tag{20}$$

This mappings are simply those of the known Voight two-index notation. Using this mapping, eqn (17) is written out in detail:

$$\begin{aligned} h_{11}Z^*_1+h_{12}Z^*_2+h_{13}Z^*_3+h_{19}Z^*_9 &= H_1, \\ h_{21}Z^*_1+h_{22}Z^*_2+h_{23}Z^*_3+h_{29}Z^*_9 &= H_2, \\ h_{31}Z^*_1+h_{32}Z^*_2+h_{33}Z^*_3+h_{39}Z^*_9 &= H_3, \\ h_{91}Z^*_1+h_{92}Z^*_2+h_{93}Z^*_3+h_{99}Z^*_9 &= H_9. \end{aligned} \tag{21}$$

After some straightforward but tedious algebraic manipulations, the simultaneous equations above for the eigenstrain Z^*_{Mn} can be solved to yield

$$\begin{aligned} Z^*_1 &= R_1\varepsilon+Q_1\sigma^0, \\ Z^*_2 &= R_2\varepsilon+Q_2\sigma^0, \\ Z^*_3 &= R_3\varepsilon+Q_3\sigma^0, \\ Z^*_9 &= R_9\varepsilon+Q_9\sigma^0, \end{aligned} \tag{22}$$

where Q_i and R_i (i = 1, 2, 3, 9) are rather lengthy and not listed here, but are tabulated in Appendix A.

4. EXAMPLES

In this section, we will examine the work-hardening behavior of a piezoelectric composite material. When shapes of inhomogeneity are elliptical, ribbon-like, rod-shaped, and penny-shaped and the material is considered to be transversely isotropic, the analytical results are presented. Let the crystalline directions of the piezoelectric material coincide with the principal axes of the inhomogeneity, and both of the piezoelectric materials of matrix and inhomogeneity be transversely isotropic with x_3 as a symmetry axis. Then, the elastic, piezoelectric and dielectric constants $\hat{C}_{iJMn}$ are expressed in a 9×9 matrix form as follows:

$$[\hat{\mathbf{C}}_{iJMn}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 & 0 & 0 & e_{31} \\ & C_{11} & C_{13} & 0 & 0 & 0 & 0 & 0 & e_{31} \\ & & C_{33} & 0 & 0 & 0 & 0 & 0 & e_{33} \\ & & & C_{44} & 0 & 0 & 0 & e_{15} & 0 \\ & & & & C_{44} & 0 & e_{15} & 0 & 0 \\ & \text{Sym.} & & & & C_{66} & 0 & 0 & 0 \\ & & & & & & -\kappa_{11} & 0 & 0 \\ & & & & & & & -\kappa_{11} & 0 \\ & & & & & & & & -\kappa_{33} \end{bmatrix}, \tag{23}$$

where $C_{66} = (C_{11} - C_{12})/2$.

Let the crystalline axes of the matrix and those of the inhomogeneities coincide, the electroelastic Eshelby's tensors for the transversely isotropic material are given below for the limiting case of an ellipsoid (Huang and Yu, 1994).

Elliptical inclusion ($a_1/a_2 = a$, $a_3 \to \infty$)

$$S_{1111} = \frac{a}{2(a+1)^2}\left\{\frac{2C_{11}+C_{12}}{C_{11}} + \frac{a+2}{a}\right\}, \quad S_{1122} = \frac{a}{2(a+1)^2}\left\{\frac{(2+a)C_{12}}{aC_{11}} - 1\right\},$$

$$S_{1133} = \frac{C_{13}}{(a+1)C_{11}}, \quad S_{1143} = \frac{e_{31}}{(a+1)C_{11}},$$

$$S_{2211} = \frac{a}{2(a+1)^2}\left\{\frac{(1+2a)C_{12}}{C_{11}} - 1\right\}, \quad S_{2222} = \frac{a}{2(a+1)^2}\left\{2a + \frac{3C_{11}+C_{12}}{C_{11}}\right\},$$

$$S_{2233} = \frac{aC_{13}}{(a+1)C_{11}}, \quad S_{2243} = \frac{ae_{31}}{(a+1)C_{11}},$$

$$S_{1212} = S_{1221} = S_{2121} = S_{2112} = \frac{a}{2(a+1)^2}\left\{\frac{a^2+a+1}{a} - \frac{C_{12}}{C_{11}}\right\},$$

$$S_{1313} = S_{1331} = S_{3131} = S_{3113} = \frac{1}{2(a+1)},$$

$$S_{2323} = S_{2332} = S_{3232} = S_{3223} = \frac{a}{2(a+1)},$$

$$S_{4141} = \frac{1}{a+1}, \quad S_{4242} = \frac{a}{a+1}. \tag{24}$$

Ribbon ($a_1 \ll a_2$, $a_3 \to \infty$)

$$S_{1111} = 1 - [a(C_{11} - C_{12})/(2C_{11})],$$

$$S_{1122} = (C_{12}/C_{11}) - [a(C_{11} + 3C_{12})/(2C_{11})],$$

$$S_{1133} = (1-a)(C_{13}/C_{11}), \quad S_{1143} = (1-a)(e_{31}/C_{11}),$$

$$S_{1212} = S_{1221} = S_{2112} = S_{2121} = 1/2 - [a(C_{11} + C_{12})/(2C_{11})],$$

$$S_{1313} = S_{1331} = S_{3113} = S_{3131} = (1-a)/2,$$

$$S_{2211} = \{(C_{12}/C_{11}) - 1\}a/2, \quad S_{2222} = a(3C_{11} + C_{12})/(2C_{11}),$$

$$S_{2233} = a(C_{13}/C_{11}), \quad S_{2243} = a(e_{31}/C_{11}),$$

$$S_{2323} = S_{2332} = S_{3223} = S_{3232} = a/2,$$
$$S_{4141} = 1-a, \quad S_{4242} = a. \tag{25}$$

Rod shape ($a_1 = a_2$, $a_3 \to \infty$)

$$S_{1111} = S_{2222} = (5C_{11}+C_{12})/(8C_{11}), \quad S_{1122} = S_{2211} = (3C_{12}-C_{11})/(8C_{11}),$$
$$S_{1133} = S_{2233} = C_{13}/(2C_{11}), \quad S_{1143} = S_{2243} = e_{31}/(2C_{11}),$$
$$S_{1212} = S_{1221} = S_{2112} = S_{2121} = (3C_{11}-C_{12})/(8C_{11}),$$
$$S_{1313} = S_{1331} = S_{3131} = S_{3113} = S_{2323} = S_{2332} = S_{3232} = S_{3223} = 1/4,$$
$$S_{4141} = S_{4242} = 1/2. \tag{26}$$

Penny shape ($a_1 = a_2 \gg a_3$, $a_3 \to 0$)

$$S_{1313} = S_{1331} = S_{3113} = S_{3131} = S_{2323} = S_{2332} = S_{3223} = S_{3232} = 1/2,$$
$$S_{1341} = S_{3141} = S_{2342} = S_{3242} = e_{15}/(2C_{44}),$$
$$S_{3311} = S_{3322} = (C_{13}\kappa_{33}+e_{31}e_{33})/(C_{33}\kappa_{33}+e_{33}^2),$$
$$S_{3333} = S_{4343} = 1, \quad S_{4311} = S_{4322} = (C_{13}e_{33}-C_{33}e_{31})/(C_{33}\kappa_{33}+e_{33}^2). \tag{27}$$

By substituting eqns (24)–(27) into eqns (18a), (18b) and (19) respectively, and adopting the generalized Voight two-index notation in eqn (20), the coefficients h_{ij} and T_{Ji} can be obtained analytically and are listed in Appendix B. Hence, eqn (18b) is written out in detail as follows:

$$H_1 = \varepsilon(T_{11}+T_{12}-2T_{13})/2+L_{13}\sigma^0,$$
$$H_2 = \varepsilon(T_{21}+T_{22}-2T_{23})/2+L_{23}\sigma^0,$$
$$H_3 = \varepsilon(T_{31}+T_{32}-2T_{33})/2+L_{33}\sigma^0,$$
$$H_9 = \varepsilon(T_{91}+T_{92}-2T_{93})/2+L_{93}\sigma^0, \tag{28}$$

where

$$L_{13} = \{2C_{13}e_{31}^*e_{31}-2C_{13}^*e_{31}^2-(C_{11}+C_{12})(e_{31}^*e_{33}+C_{13}^*\kappa_{33}) + (C_{11}^*+C_{12}^*)(e_{31}e_{33}+C_{13}\kappa_{33})\}/L,$$

$$L_{23} = \{2C_{13}e_{31}^*e_{31}-2C_{13}^*e_{31}^2-(C_{11}+C_{12})(e_{31}^*e_{33}+C_{13}^*\kappa_{33}) + (C_{11}^*+C_{12}^*)(e_{31}e_{33}+C_{13}\kappa_{33})\}/L,$$

$$L_{33} = 1-\{[-2C_{13}e_{33}^*e_{31}+2C_{33}^*e_{31}^2+(C_{11}+C_{12})(e_{33}^*e_{33}+C_{33}^*\kappa_{33}) - 2C_{13}^*(e_{31}e_{33}+C_{13}\kappa_{33})]/L\},$$

$$L_{93} = \{-2e_{33}^*e_{31}^2+2e_{31}^*e_{31}e_{33}-(C_{11}+C_{12})(e_{33}^*\kappa_{33}-\kappa_{33}^*e_{33})+2C_{13}(e_{31}^*\kappa_{33}-e_{31}\kappa_{33}^*)\}/L,$$
$$L = 2C_{33}e_{31}^2-4C_{13}e_{31}e_{33}+(C_{11}+C_{12})(e_{33}^2+C_{33}\kappa_{33})-2C_{13}^2\kappa_{33}. \tag{29}$$

Finally, the internal stresses σ_{ij} in eqn (12) are evaluated by substituting eqn (22) into eqn (14) as

$$\sigma^0 = \frac{\sigma_y + f(Y + R_1 Y_1 + R_2 Y_2 + R_3 Y_3 + R_9 Y_9)\varepsilon}{1 - f(Q_1 Y_1 + Q_2 Y_2 + Q_3 Y_3 + Q_9 Y_9)}, \tag{30}$$

where Q_i and R_i ($i = 1, 2, 3, 9$) are listed in Appendix A, and the constants Y and Y_i are given below.

For an elliptical inhomogeneity oriented with its generatrix parallel to the x_3 axis,

$$\begin{aligned} Y &= \{C_{11}^2 - C_{12}^2 + 4(C_{12}C_{13} - C_{11}C_{13} - C_{13}^2 + C_{11}C_{33})\}/(4C_{11}), \\ Y_1 &= a(C_{12} - C_{11})(2C_{13} - C_{11} - C_{12})/\{2(1+a)C_{11}\}, \\ Y_2 &= (C_{12} - C_{11})(2C_{13} - C_{11} - C_{12})/\{2(1+a)C_{11}\}, \\ Y_3 &= (C_{11}C_{13} - C_{12}C_{13} + 2C_{13}^2 - 2C_{11}C_{33})/(2C_{11}), \\ Y_9 &= (C_{11}e_{31} - C_{12}e_{31} + 2C_{13}e_{31} - 2C_{11}e_{33})/(2C_{11}). \end{aligned} \tag{31}$$

For a ribbon-like inhomogeneity,

$$\begin{aligned} Y &= \{C_{11}^2 - C_{12}^2 + 4(C_{12}C_{13} - C_{11}C_{13} - C_{13}^2 + C_{11}C_{33})\}/(4C_{11}), \\ Y_1 &= a(C_{12} - C_{11})(2C_{13} - C_{11} - C_{12})/(2C_{11}), \\ Y_2 &= (1-a)(C_{12} - C_{11})(2C_{13} - C_{11} - C_{12})/(2C_{11}), \\ Y_3 &= (C_{11}C_{13} - C_{12}C_{13} + 2C_{13}^2 - 2C_{11}C_{33})/(2C_{11}), \\ Y_9 &= (C_{11}e_{31} - C_{12}e_{31} + 2C_{13}e_{31} - 2C_{11}e_{33})/(2C_{11}). \end{aligned} \tag{32}$$

For a rod-shaped inhomogeneity with its axes parallel to the tensile direction,

$$\begin{aligned} Y &= \{C_{11}^2 - C_{12}^2 + 4(C_{12}C_{13} - C_{11}C_{13} - C_{13}^2 + C_{11}C_{33})\}/(4C_{11}), \\ Y_1 &= Y_2 = (C_{12} - C_{11})(2C_{13} - C_{11} - C_{12})/(4C_{11}), \\ Y_3 &= \{(C_{11} - C_{12} + 2C_{13})C_{13} - 2C_{11}C_{33}\}/(2C_{11}), \\ Y_9 &= \{(C_{11} - C_{12} + 2C_{13})e_{31} - 2C_{11}e_{33}\}/(2C_{11}). \end{aligned} \tag{33}$$

For a penny-shaped inhomogeneity with flat faces perpendicular to the tensile direction,

$$\begin{aligned} Y = Y_1 = Y_2 &= \{2C_{33}e_{31}^2 - 4C_{13}e_{31}e_{33} + (C_{11} + C_{12})e_{33}^2 \\ &\quad + (C_{11}C_{33} - 2C_{13}^2 + C_{12}C_{33})\kappa_{33}\}/\{2(e_{33}^2 + C_{33}\kappa_{33})\}, \\ Y_3 = Y_9 &= 0. \end{aligned} \tag{34}$$

The equations above show that Y and Y_i are independent on the size of the inhomogeneities but are dependent on the shape.

The yielding stress of this piezoelectric composite material is

$$\frac{\sigma_y}{1 - f(Q_1 Y_1 + Q_2 Y_2 + Q_3 Y_3 + Q_9 Y_9)}, \tag{35}$$

and the linear work-hardening rate is

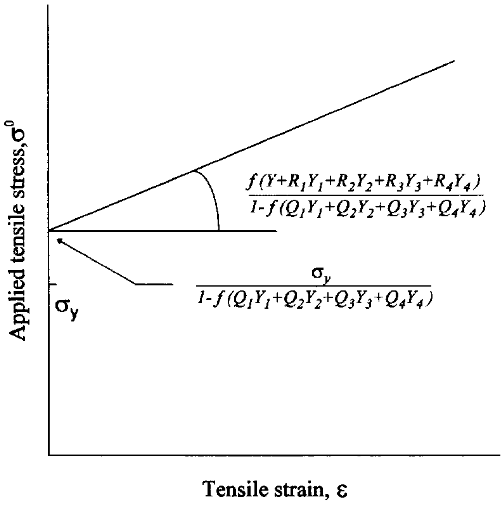

Fig. 1. A schematic representation of the macroscopic stress strain relation predicted by eqn (36) for a piezoelectric composite material.

$$\frac{d\sigma^0}{d\varepsilon}=\frac{f(Y+R_1Y_1+R_2Y_2+R_3Y_3+R_9Y_9)}{1-f(Q_1Y_1+Q_2Y_2+Q_3Y_3+Q_9Y_9)}. \tag{36}$$

By inspection of eqns (35) and (36) it is concluded that the yielding stress and linear work-hardening rate of piezoelectric composite materials are also independent on the size of the inhomogeneities but are dependent on the shape.

A schematic representation of the macroscopic stress strain relation of eqn (36) is depicted in Fig. 1. As an example of PZT-5 matrix with PZT-4 penny-shaped and rod-shaped inhomogeneities, respectively, the work-hardening rate of the piezoelectric composite material for different values of volume fraction is plotted in Fig. 2. The elastic, dielectric, and piezoelectric properties of the constituents used in the numerical computations are given in the following:

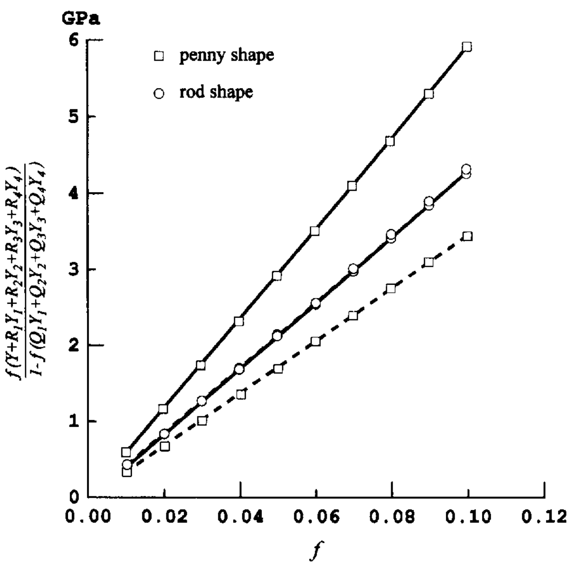

Fig. 2. The work-hardening rates of the piezoelectric composite material for different values of volume fraction (solid lines for piezoelectric inhomogeneities, dashed lines for the inhomogeneities without a piezoelectric coupling effect).

PZT-4 inhomogeneities

$$C^*_{11} = 139\,\text{GPa},\quad C^*_{12} = 77.8\,\text{GPa},\quad C^*_{13} = 74.3\,\text{GPa},\quad C^*_{33} = 115\,\text{GPa},$$

$$C^*_{44} = 25.6\,\text{GPa},\quad e^*_{31} = -5.2\,\text{c/m}^2,\quad e^*_{33} = 15.1\,\text{c/m}^2,\quad e^*_{15} = 12.7\,\text{c/m}^2,$$

$$\kappa^*_{11} = 6.4605\times10^{-9}\,\text{c}^2/\text{N m}^2,\quad \kappa^*_{33} = 5.6198\times10^{-9}\,\text{c}^2/\text{N m}^2.$$

PZT-5 matrix

$$C_{11} = 121\,\text{GPa},\quad C_{12} = 75.4\,\text{GPa},\quad C_{13} = 75.2\,\text{GPa},\quad C_{33} = 111\,\text{GPa},$$

$$C_{44} = 21.1\,\text{GPa},\quad e_{31} = -5.4\,\text{c/m}^2,\quad e_{33} = 15.8\,\text{c/m}^2,\quad e_{15} = 12.3\,\text{c/m}^2,$$

$$\kappa_{11} = 8.1066\times10^{-9}\,\text{c}^2/\text{N m}^2,\quad \kappa_{33} = 7.3455\times10^{-9}\,\text{c}^2/\text{N m}^2.$$

As indicated in Fig. 2, the piezoelectric composite material shows linear work-hardening although the matrix is perfectly plastic. The linear work-hardening rate is proportional to the volume fraction of the inhomogeneities and are dependent on the shape. It is also shown in Fig. 2 that when piezoelectric coupling is absent (i.e. $e_{imn} = 0$), the material also behaves as a linearly hardening material. As is clear from the figure, the linear work-hardening rate of the piezoelectric composite with penny-shaped inhomogeneities is higher than that of the material without a piezoelectric coupling effect. However, in the case of rod-shaped inhomogeneities, the linear work-hardening rates for the material with and without an electromechanical coupling effect are nearly equal over the range of volume fraction.

5. CONCLUSIONS

This work has been concerned with the development of an analytical approach to predict the work-hardening behavior of composite material subjected to a uniaxial tension. The analytical approach that has been forwarded is based on the variational principle and the equivalent inclusion method. It is concluded that the mechanical behavior of the piezoelectric composite materials is controlled by the nature of the inhomogeneities and the matrix. When both the matrix and the inhomogeneities of piezoelectric materials are transversely isotropic with different electroelastic moduli, the linear work-hardening rate is predicted and it is obtained in closed form when shapes of the inhomogeneities are elliptical, ribbon-like, rod-shaped, and penny-shaped. The work-hardening rate is proportional to the volume fraction of the inhomogeneities. Moreover, the results show that the yielding stress and the work-hardening rate are independent on the size of the inhomogeneities but are dependent on the shape of the inhomogeneities.

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APPENDIX A

$$Q_9 = \{[(\mu\beta_3-\alpha\beta_2)L_{23}+(\alpha L_{93}-\mu L_{33})\beta_1]h_{11}+[\alpha(\beta_2h_{21}-\beta_1h_{91})+\mu(\beta_1h_{31}-\beta_3h_{21})]L_{13}\}/(\alpha\omega-\lambda\mu),$$

$$Q_3 = \{\beta_1h_{11}L_{33}+(\beta_3h_{21}-\beta_1h_{31})L_{13}-\beta_3h_{11}L_{23}-\lambda Q_9\}/\alpha,$$

$$Q_2 = \{h_{11}L_{23}-h_{21}L_{13}-(h_{11}h_{23}-h_{13}h_{21})Q_3-(h_{11}h_{29}-h_{19}h_{21})Q_9\}/\beta_1,$$

$$Q_1 = (L_{13}-h_{12}Q_2-h_{13}Q_3-h_{19}Q_9)/h_{11},$$

$$R_9 = \{(\mu\beta_3-\alpha\beta_2)(T_{21}+T_{22}-2T_{23})h_{11}+[\alpha(T_{91}+T_{92}-2T_{93})-\mu(T_{31}+T_{32}-2T_{33})]$$
$$\cdot\beta_1h_{11}+[\alpha(\beta_2h_{21}-\beta_1h_{91})+\mu(\beta_1h_{31}-\beta_3h_{21})](T_{11}+T_{12}-2T_{13})\}/2(\alpha\omega-\lambda\mu),$$

$$R_3 = \{[\beta_1(T_{31}+T_{32}-2T_{33})-\beta_3(T_{21}+T_{22}-2T_{23})]h_{11}+(\beta_3h_{21}-\beta_1h_{31})\cdot(T_{11}+T_{12}-2T_{13})-2\lambda R_9\}/2\alpha,$$

$$R_2 = \{(T_{21}+T_{22}-2T_{23})h_{11}-(T_{11}+T_{12}-2T_{13})h_{21}-2(h_{11}h_{23}-h_{13}h_{21})R_3-2(h_{11}h_{29}-h_{19}h_{21})R_9\}/2\beta_1,$$

$$R_1 = \{(T_{11}+T_{12}-2T_{13})-2h_{12}R_2-2h_{13}R_3-2h_{19}R_9\}/2h_{11}, \quad \text{(A1)}$$

where

$$\alpha = \beta_1(h_{11}h_{33}-h_{13}h_{31})-(h_{11}h_{23}-h_{13}h_{21})\beta_3,$$
$$\mu = \beta_1(h_{11}h_{93}-h_{13}h_{91})-(h_{11}h_{23}-h_{13}h_{21})\beta_2,$$
$$\lambda = \beta_1(h_{11}h_{39}-h_{19}h_{31})-(h_{11}h_{29}-h_{19}h_{21})\beta_3,$$
$$\omega = \beta_1(h_{11}h_{99}-h_{19}h_{91})-(h_{11}h_{29}-h_{19}h_{21})\beta_2,$$
$$\beta_1 = h_{11}h_{22}-h_{12}h_{21},$$
$$\beta_2 = h_{11}h_{92}-h_{12}h_{91},$$
$$\beta_3 = h_{11}h_{32}-h_{12}h_{31}. \quad \text{(A2)}$$

APPENDIX B

Elliptical inclusion ($a_1/a_2 = a$, $a_3 \to \infty$)

$$h_{11} = \{(2C^*_{11}+3aC^*_{11}-aC^*_{12}+aC_{11}+2a^2C_{11})C_{11}+a(C^*_{11}+C^*_{12}+2aC^*_{12})C_{12}-a(1+2a)C^2_{12}\}/\{2(1+a)^2C_{11}\},$$

$$h_{12} = C_{12}+\{(C^*_{11}-C_{11})[(2+a)C_{12}-aC_{11}]+(C^*_{12}-C_{12})[(2a^2+3a)C_{11}+aC_{12}]\}/2(1+a)^2C_{11},$$

$$h_{13} = C_{13}+\{(C^*_{11}-C_{11})+a(C^*_{12}-C_{12})\}C_{13}/(1+a)C_{11},$$

$$h_{19} = e_{31}+\{(C^*_{11}-C_{11})+a(C^*_{12}-C_{12})\}e_{31}/(1+a)C_{11},$$

$$h_{21} = \{(2C^*_{12}-aC^*_{11}+3aC^*_{12}+aC_{11})C_{11}+a(C^*_{11}+2aC^*_{11}+C^*_{12})C_{12}-aC^2_{12}\}/\{2(1+a)^2C_{11}\},$$

$$h_{22} = C_{11}+\{(C^*_{11}-C_{11})[(2a^2+3a)C_{11}+aC_{12}]+(C^*_{12}-C_{12})[(2+a)C_{12}-aC_{11}]\}/2(1+a)^2C_{11},$$

$$h_{23} = C_{13}+\{a(C^*_{11}-C_{11})+(C^*_{12}-C_{12})\}C_{13}/(1+a)C_{11},$$

$$h_{29} = e_{31}+\{a(C^*_{11}-C_{11})+(C^*_{12}-C_{12})\}e_{31}/(1+a)C_{11},$$

$$h_{31} = \{(C_{11}+aC_{12})C^*_{13}+a(C_{11}-C_{12})C_{13}\}/\{(1+a)C_{11}\},$$

$$h_{32} = C_{13}+(C^*_{13}-C_{13})(aC_{11}+C_{12})/(1+a)C_{11},$$

$$h_{33} = C_{33}+(C^*_{13}-C_{13})C_{13}/C_{11},$$

$$h_{39} = e_{33}+(C^*_{13}-C_{13})e_{31}/C_{11},$$

$$h_{91} = \{(C_{11}+aC_{12})e^*_{31}+a(C_{11}-C_{12})e_{31}\}/\{(1+a)C_{11}\},$$

$$h_{92} = e_{31}+(e^*_{31}-e_{31})(aC_{11}+C_{12})/(1+a)C_{11},$$

$$h_{93} = e_{33}+(e^*_{31}-e_{31})C_{13}/C_{11},$$

$$h_{99} = -k_{33}+(e^*_{31}-e_{31})e_{31}/C_{11}, \quad \text{(B1)}$$

and

$$T_{11}=\frac{a\{(C_{11}-C^*_{11}-2aC^*_{11}-C^*_{12}+2aC_{11})C_{11}+(C^*_{11}+C^*_{12}+2aC^*_{12})C_{12}-(1+2a)C^2_{12}\}}{2(1+a)^2C_{11}},$$

$$T_{12}=\{(aC_{11}-aC^*_{11}-2C^*_{12}-aC^*_{12})C_{11}+(2C^*_{11}+aC^*_{11}+aC^*_{12}-aC_{12})C_{12}\}/\{2(1+a)^2C_{11}\},$$

$$T_{13}=\{(C^*_{11}+aC_{11}+aC^*_{12}-aC_{12})C_{13}-(1+a)C^*_{13}C_{11}\}/\{(1+a)C_{11}\},$$

$$T_{21}=a\{(C_{11}-C^*_{11}-C^*_{12}-2aC^*_{12})C_{11}+(C^*_{11}+2aC^*_{11}+C^*_{12}-C_{12})C_{12}\}/\{2(1+a)^2C_{11}\},$$

$$T_{22}=\{[(a+2)(C_{11}-C^*_{11})-aC^*_{12}]C_{11}+[(a+2)(C^*_{12}-C_{12})+aC^*_{11}]C_{12}\}/\{2(1+a)^2C_{11}\},$$

$$T_{23}=\{(aC^*_{11}+C_{11}+C^*_{12}-C_{12})C_{13}-(1+a)C^*_{13}C_{11}\}/\{(1+a)C_{11}\},$$

$$T_{31}=a(C_{12}-C_{11})(C^*_{13}-C_{13})/\{(1+a)C_{11}\},$$

$$T_{32}=(C_{12}-C_{11})(C^*_{13}-C_{13})/\{(1+a)C_{11}\},$$

$$T_{33}=\{(C^*_{13}-C_{13})C_{13}-(C^*_{33}-C_{33})C_{11}\}/C_{11},$$

$$T_{91}=a(C_{12}-C_{11})(e^*_{31}-e_{31})/\{(1+a)C_{11}\},$$

$$T_{92}=(C_{12}-C_{11})(e^*_{31}-e_{31})/\{(1+a)C_{11}\},$$

$$T_{93}=\{(e^*_{31}-e_{31})C_{13}-(e^*_{33}-e_{33})C_{11}\}/C_{11}. \qquad \text{(B2)}$$

Ribbon ($a_1 \ll a_2, a_3 \to \infty$)

$$h_{11}=\{(2C^*_{11}-aC^*_{11}-aC^*_{12}+aC_{11})C_{11}+a(C^*_{11}+C^*_{12})C_{12}-aC^2_{12}\}/(2C_{11}),$$

$$h_{12}=C_{12}+\{(C^*_{11}-C_{11})[(2-3a)(C_{12}-aC_{11})]+a(C^*_{12}-C_{12})(3C_{11}+C_{12})\}/(2C_{11}),$$

$$h_{13}=C_{13}+\{(1-a)(C^*_{11}-C_{11})+a(C^*_{12}-C_{12})\}C_{13}/C_{11},$$

$$h_{19}=e_{31}+\{(1-a)(C^*_{11}-C_{11})+a(C^*_{12}-C_{12})\}e_{31}/C_{11},$$

$$h_{21}=\{(2C^*_{12}-aC^*_{11}-aC^*_{12}+aC_{11})C_{11}+a(C^*_{11}+C^*_{12})C_{12}-aC^2_{12}\}/(2C_{11}),$$

$$h_{22}=C_{11}+\{a(C^*_{11}-C_{11})(3C_{11}+C_{12})+(C^*_{12}-C_{12})[(2-3a)C_{12}-aC_{11}]\}/(2C_{11}),$$

$$h_{23}=C_{13}+\{a(C^*_{11}-C_{11})+(1-a)(C^*_{12}-C_{12})\}C_{13}/C_{11},$$

$$h_{29}=e_{31}+\{a(C^*_{11}-C_{11})+(1-a)(C^*_{12}-C_{12})\}e_{31}/C_{11},$$

$$h_{31}=\{(C_{11}-aC_{11}+aC_{12})C^*_{13}+a(C_{11}-C_{12})C_{13}\}/C_{11},$$

$$h_{32}=C_{13}+(C^*_{13}-C_{13})\{2aC_{11}+2(1-a)C_{12}\}/(2C_{11}),$$

$$h_{33}=C_{33}+(C^*_{13}-C_{13})C_{13}/C_{11},$$

$$h_{39}=e_{33}+(C^*_{13}-C_{13})e_{31}/C_{11},$$

$$h_{91}=\{(C_{11}-aC_{11}+aC_{12})e^*_{31}+a(C_{11}-C_{12})e_{31}\}/C_{11},$$

$$h_{92}=e_{31}+(e^*_{31}-e_{31})\{2aC_{11}+2(1-a)C_{12}\}/(2C_{11}),$$

$$h_{93}=e_{33}+(e^*_{31}-e_{31})C_{13}/C_{11},$$

$$h_{99}=-\kappa_{33}+(e^*_{31}-e_{31})e_{31}/C_{11}, \qquad \text{(B3)}$$

and

$$T_{11}=a(C^*_{11}+C^*_{12}-C_{11}-C_{12})(C_{12}-C_{11})/(2C_{11}),$$

$$T_{12}=\{(C^*_{11}-C_{11})(2C_{12}-3aC_{12}-aC_{11})+(C^*_{12}-C_{12})(3aC_{11}-2C_{11}+aC_{12})\}/(2C_{11}),$$

$$T_{13}=\{(C^*_{11}-aC^*_{11}+aC^*_{12}+aC_{11}-aC_{12})C_{13}-C_{11}C^*_{13}\}/C_{11},$$

$$T_{21}=a(C^*_{11}+C^*_{12}-C_{11}-C_{12})(C_{12}-C_{11})/(2C_{11}),$$

$$T_{22}=\{(C^*_{12}-C_{12})(2C_{12}-3aC_{12}-aC_{11})+(C^*_{11}-C_{11})(3aC_{11}-2C_{11}+aC_{12})\}/(2C_{11}),$$

$$T_{23}=\{[a(C^*_{11}-C_{11})+(1-a)(C^*_{12}-C_{12})]C_{13}-(C^*_{13}-C_{13})C_{11}\}/C_{11},$$

$$T_{31}=a(C_{12}-C_{11})(C^*_{13}-C_{13})/C_{11},$$

$$T_{32}=(1-a)(C_{11}-C_{12})(C_{13}-C^*_{13})/C_{11},$$

$$T_{33}=\{(C^*_{13}-C_{13})C_{13}-(C^*_{33}-C_{33})C_{11}\}/C_{11},$$

$$T_{91}=a(C_{12}-C_{11})(e^*_{31}-e_{31})/C_{11},$$

$$T_{92}=(1-a)(C_{11}-C_{12})(e_{31}-e^*_{31})/C_{11},$$

$$T_{93}=\{(e^*_{31}-e_{31})C_{13}-(e^*_{33}-e_{33})C_{11}\}/C_{11}. \qquad \text{(B4)}$$

Rod shape ($a_1 = a_2$, $a_3 \to \infty$)

$$h_{11} = h_{22} = C_{11} + \{(C^*_{11} - C_{11})(5C_{11} + C_{12}) + (C^*_{12} - C_{12})(3C_{12} - C_{11})\}/(8C_{11}),$$
$$h_{12} = h_{21} = C_{12} + \{(C^*_{11} - C_{11})(3C_{12} - C_{11}) + (C^*_{12} - C_{12})(5C_{11} + C_{12})\}/(8C_{11}),$$
$$h_{13} = h_{23} = C_{13} + \{(C^*_{11} - C_{11}) + (C^*_{12} - C_{12})\}C_{13}/(2C_{11}),$$
$$h_{19} = h_{29} = e_{31} + \{(C^*_{11} - C_{11}) + (C^*_{12} - C_{12})\}e_{31}/(2C_{11}),$$
$$h_{31} = h_{32} = C_{13} + (C^*_{13} - C_{13})(C_{11} + C_{12})/(2C_{11}),$$
$$h_{33} = C_{33} + (C^*_{13} - C_{13})C_{13}/C_{11},$$
$$h_{39} = e_{33} + (C^*_{13} - C_{13})e_{31}/(2C_{11}),$$
$$h_{91} = h_{92} = e_{31} + (e^*_{31} - e_{31})(C_{11} + C_{12})/(2C_{11}),$$
$$h_{93} = e_{33} + (e^*_{31} - e_{31})C_{13}/C_{11},$$
$$h_{99} = -\kappa_{33} + (e^*_{31} - e_{31})e_{31}/C_{11}, \quad \text{(B5)}$$

and

$$T_{11} = T_{22} = \{(C^*_{11} - C_{11})(C_{12} - 3C_{11}) + (C^*_{12} - C_{12})(3C_{12} - C_{11})\}/(8C_{11}),$$
$$T_{12} = T_{21} = \{(C_{11} - C^*_{11} - 3C^*_{12})C_{11} + (3C^*_{11} + C^*_{12} - C_{12})C_{12}\}/(8C_{11}),$$
$$T_{13} = T_{23} = \{(C^*_{11} + C_{11} + C^*_{12} - C_{12})C_{13} - 2C^*_{13}C_{11}\}/(2C_{11}),$$
$$T_{31} = T_{32} = (C^*_{13} - C_{13})(C_{12} - C_{11})/(2C_{11}),$$
$$T_{33} = \{(C^*_{13} - C_{13})C_{13} - (C^*_{33} - C_{33})C_{11}\}/C_{11},$$
$$T_{91} = T_{92} = (e^*_{31} - e_{31})(C_{12} - C_{11})/(2C_{11}),$$
$$T_{93} = \{(e^*_{31} - e_{31})C_{13} - (e^*_{33} - e_{33})C_{11}\}/C_{11}. \quad \text{(B6)}$$

Penny shape ($a_1 = a_2 \gg a_3$, $a_3 \to 0$):

$$h_{11} = h_{22} = C_{11} + \{(e^*_{31} - e_{31})(C_{13}e_{33} - C_{33}e_{31}) + (C^*_{13} - C_{13})(C_{13}\kappa_{33} + e_{31}e_{33})\}/P_1,$$
$$h_{12} = h_{21} = C_{12} + \{(e^*_{31} - e_{31})(C_{13}e_{33} - C_{33}e_{31}) + (C^*_{13} - C_{13})(C_{13}\kappa_{33} + e_{31}e_{33})\}/P_1,$$
$$h_{13} = h_{23} = C^*_{13},$$
$$h_{19} = h_{29} = e^*_{31},$$
$$h_{31} = h_{32} = C_{13} + \{(e^*_{33} - e_{33})(C_{13}e_{33} - C_{33}e_{31}) + (C^*_{33} - C_{33})(C_{13}\kappa_{33} + e_{31}e_{33})\}/P_1,$$
$$h_{33} = C^*_{33},$$
$$h_{39} = h_{93} = e^*_{33},$$
$$h_{91} = h_{92} = e_{31} + \{(\kappa_{33} - \kappa^*_{33})(C_{13}e_{33} - C_{33}e_{311}) + (e^*_{33} - e_{33})(C_{13}\kappa_{33} + e_{31}e_{33})\}/P_1,$$
$$h_{99} = -\kappa_{33},$$
$$P_1 = C_{33}\kappa_{33} + e^2_{33}, \quad \text{(B7)}$$

and

$$T_{11} = T_{22} = \{[(C^*_{13} - C_{13})(e_{31}e_{33} + C_{13}\kappa_{33}) - (e^*_{31} - e_{31})(C_{33}e_{31} - C_{13}e_{33})]/(e^2_{33} + C_{33}\kappa_{33})\} - (C^*_{11} - C_{11}),$$
$$T_{12} = T_{21} = \{[(C^*_{13} - C_{13})(e_{31}e_{33} + C_{13}\kappa_{33}) - (e^*_{31} - e_{31})(C_{33}e_{31} - C_{13}e_{33})]/(e^2_{33} + C_{33}\kappa_{33})\} - (C^*_{12} - C_{12}),$$
$$T_{13} = T_{23} = T_{33} = T_{93} = 0,$$
$$T_{31} = T_{32} = \{(C_{13}e_{33} - C_{33}e_{31})e^*_{33} + (C^*_{33}e_{31} - C^*_{13}e_{33})e_{33} + (C^*_{33}C_{13} - C^*_{13}C_{33})\kappa_{33}\}/(e^2_{33} + C_{33}\kappa_{33}),$$
$$T_{91} = T_{92} = \{(C_{33}e_{31} - C_{13}e_{33})\kappa^*_{33} + (e^*_{33}e_{31} - e^*_{31}e_{33})e_{33} + (C_{13}e^*_{33} - C_{33}e^*_{31})\kappa_{33}\}/(e^2_{33} + C_{33}\kappa_{33}). \quad \text{(B8)}$$